

# Constraints and Solutions of Quantum Gravity in Metric Representation

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## Abstract

We construct the regularised Wheeler–De Witt operator demanding that the algebra of constraints of quantum gravity is anomaly free. We find that for a subset of all wavefunctions being integrals of scalar densities this condition can be satisfied. We proceed to finding exact solutions of quantum gravity being of the form of functionals of volume and average curvature of compact three-manifold.

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# 1 Introduction

One of the outstanding problems of modern theoretical physics is the construction of quantum theory of gravity [1]. Indeed, it has been claimed many times that various unsolved problems like the cosmological constant problem, the problem of origin of the universe, the problem of black holes radiation will find their ultimate solution once this theory is finally constructed and properly understood. Some [2], claim that the theory of quantum gravity will also shed some light on the fundamental problems of quantum mechanics and even on the origin of mind. These all prospects are very exciting indeed, however, up to now, the shapes of the future theory are still very obscured.

Nowadays there are two major ways of approaching the problem of quantum gravity. The first one is associated with the broad term ‘superstrings’. In this approach the starting point is a two-dimensional quantum field theory which yields quantum gravity as a part of the resulting low-energy effective theories. It is clear that in superstrings, like in other, less developed approaches in whose gravity appears as an effective theory, it does not make sense to try to “quantize” classical gravity.

In the canonical approach one does something opposite: the idea is to pick up some structures which appear already at the classical level and then promote them to define the quantum theory. In both the standard canonical approach in metric representation, which we will follow here, and in the approach based on loop variables [3], these fundamental structures are constraints of the classical canonical formalism reflecting the symmetries and dynamics of the theory, and their algebra. There are good reasons for such an approach. The equivalence principle is the main physical principle behind the classical theory of gravity; this principle leads to the general co-ordinate invariance and selects the Einstein–Hilbert action as the simplest possible one.

Another building block of quantum theory is the quantization procedure. Here one encounters the problem as to if a generalisation of the standard Dirac procedure of quantization of gauge theories in hamiltonian language is necessary. This would be the case if one shows that the standard approach is not capable of producing any interesting results. It is not excluded that this may be eventually the result of possible failure of investigations using standard techniques, however, in our opinion, at the moment there is no reason to modify the basic principles of quantum theory.

Our starting point consists therefore of

- (i) The classical constraints of Einstein's gravity: the diffeomorphism constraint generating diffeomorphism of the spatial three-surface "of constant time"

$$\mathcal{D}_a = \nabla_b \pi^{ab} \quad (1)$$

and the hamiltonian constraint generating "pushes in time direction":

$$\mathcal{H} = \kappa^2 G_{abcd} \pi^{ab} \pi^{cd} - \frac{1}{\kappa^2} \sqrt{h} (R + 2\Lambda) \quad (2)$$

In the formulas above  $\pi^{ab}$  are momenta associated with the three-metric  $h_{ab}$ ,

$$G_{abcd} = \frac{1}{2\sqrt{h}} (h_{ac} h_{bd} + h_{ad} h_{bc} - h_{ab} h_{cd})$$

is the Wheeler–De Witt metric,  $R$  is the three-dimensional curvature scalar,  $\kappa$  is the gravitational constant, and  $\Lambda$  the cosmological constant. The constraints satisfy the following Poisson bracket algebra

$$[\mathcal{D}, \mathcal{D}] \sim \mathcal{D}, \quad (3)$$

$$[\mathcal{D}, \mathcal{H}] \sim \mathcal{H}, \quad (4)$$

$$[\mathcal{H}, \mathcal{H}] \sim \mathcal{D}. \quad (5)$$

- (ii) The rules of quantization given by the metric representation of the canonical commutational relations

$$[\pi^{ab}(x), h_{cd}(y)] = -i \delta_c^{(a} \delta_d^{b)} \delta(x, y),$$

$$\pi^{ab}(x) = -i \frac{\delta}{\delta h_{ab}(x)}.$$

- (iii) The Dirac procedure according to which one imposes constraints quantum mechanically by demanding that they (or, better, the corresponding operators) annihilate the subspace of the Hilbert space called the set of physical states. Bearing in mind the notorious regularisation and renormalization problems of quantum field theory, one should clearly state what the phrase "corresponding operators" means. In general, different choices of such operators could result in different quantum theories.

In the canonical approach, the points (i) to (iii) above encompass the whole of the input in our disposal in construction of the quantum theory. In particular, we do not know what is the correct physical inner product, and thus we do not know if the relevant operators are hermitean or not. Besides, we do not even know if, in the case of quantum gravity, we should demand these operators to be hermitean: the hamiltonian annihilates the physical states (the famous time problem [4]) and thus unitary evolution does not play any privileged role anymore. It follows that, perhaps, we cannot distinguish “relevant” wave functions by demanding that they are normalizable, as in the case of quantum mechanics, in fact, since the probabilistic interpretation of the “wavefunction of the universe” is doubtful, it is not clear at all if the norm of this wavefunction is to be 1.

In the recent paper [9] a class of exact solutions of the Wheeler–De Witt equation was found. In that paper we used the heat kernel to regularise the hamiltonian operator and inserted the particular operator ordering. The question arises what is the level of arbitrariness in this construction. In other words, could we construct other (possibly simpler) regularised hamiltonian operators and what would be their properties? This question is the subject of the present paper.

It is clear from the discussion above that the only principle, we can base our construction on is the principle that the algebra of constraints is to be anomaly-free, that is, the corresponding algebra of commutators of quantum constraints is weakly identical with the classical one. This means that the structure of the Poisson bracket algebra (3–5) is to be preserved, in the sense which will be explained below, on the quantum level. The following section is devoted to the analysis of this problem. In section 3 we investigate solutions of the resulting equations. Some more technical results are presented in the Appendix.

## 2 The commutator algebra and construction of regularised operators

As explained in Introduction, our starting point in construction of the quantum hamiltonian operator (the Wheeler–De Witt operator) is the algebra

(3–5) and we demand that the same algebra holds on the quantum level. At this point one should ask the question why we impose such a condition. One of the possible answers is that the closure of the algebra is the only principle which makes it possible to find operators corresponding to classical constraints, required by the Dirac procedure. Another argument to be found in the literature is that if the algebra is anomalous (that is, if there are additional terms resulting from the commutators of the constraint operators) one

cannot find any solutions of the quantum constraints. This does not apply here since we know explicitly that for a particular regularisation/regularisation prescription introduced in [9] a class of solutions exist, and on solutions the algebra closes identically. The above argument can be therefore rephrased as follows. We want the algebra to close because if it does not, then the solutions we find will have to be in the kernel of the anomaly (we know that the kernel is non-empty because solutions do exist) which would mean that the conditions we impose will be more restrictive than the ones imported from the classical theory. This by itself is not a disaster, since, in any case, the classical limit will be the same, but we would like to depart from the classical theory as little as we possibly could.

From our point of view there is another important argument in favour of preserving the constraint algebra structure. The vanishing of anomaly is, as it turns out, a quite restrictive condition which makes it possible to restrict the form of the employed regulator. The idea is therefore to find a regularisation/renormalization procedure consistent with the symmetries dictated by the classical general relativity and to restrict it further by demanding that there are solutions of the theory in a class of natural wavefunctions.

The problem of commutator algebra has been analysed in [5], [6] (and recently in [7]) with the result that formal manipulations involving point splitting lead to ambiguous final expressions. This conclusion is hardly surprising: It is well known [8] that to compute an anomaly one should first define the space of states on which the operator in question act. Then one should clearly state what is the procedure of extracting the finite part of formally divergent expressions. Thus the right question to ask is *not* what is the formal commutator of constraints but: Given a space of states, does it exist a regularisation/renormalization prescription such that the renormalized action of the operators on the states closes? It should be stressed that this question is based on the basic physical interpretation of the rele-

vant operators; indeed, the constraints operators are generators of physical transformations which make sense only in terms of results of their action on appropriate states. As it will be seen below, the condition guaranteeing the absence of anomalies, not surprisingly, becomes different when the operators act on different states. The technical reason is simply that different states pick up different parts of the regulator. This fact is, of course, well known in investigation of anomalies in quantum field theory in canonical quantization language (cf. [8]).

As stressed above, we must start with choosing the initial space of wave functionals. We assume that this space of states is the space of functions of Riemannian functionals, i.e., integrals over compact three-space  $M$  of scalar densities built of polynomials in Ricci tensor, like  $\mathcal{V} = \int_M \sqrt{h}$ ,  $\mathcal{R} = \int_M \sqrt{h} R$ , etc.:

$$\Psi = \Psi(\mathcal{V}, \mathcal{R}, \dots). \quad (6)$$

We choose the following representation of the diffeomorphism constraint

$$\mathcal{D}_a(x) = -i \nabla_b^x \frac{\delta}{\delta h_{ab}(x)}, \quad (7)$$

where we employed the notation  $\nabla_b^x$  meaning that the covariant derivative acts at the point  $x$ . Then we see that diffeomorphism constraint annihilates all the states and the commutator relation (3) is identically satisfied. This is the reason for a particular, natural ordering in (7). Moreover we see that the relation (4) reduces to the *formal* relation

$$\mathcal{D}(\mathcal{H}\Psi) \sim \mathcal{H}\Psi. \quad (8)$$

Now we must turn to the heart of the problem, the construction of the Wheeler–De Witt operator. It is well known that second functional derivative acting at the same point on a local functional produces divergent result. We deal with this problem by making the point split in the kinetic term, to wit

$$G_{abcd}(x) \pi^{ab}(x) \pi^{cd}(x) \implies \int dx' K_{abcd}(x, x'; t) \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')},$$

where  $K_{abcd}(x, x'; t)$  satisfies

$$\lim_{t \rightarrow 0^+} K_{abcd}(x, x'; t) = G_{abcd}(x') \delta(x, x').$$

By virtue of the correspondence principle, we take

$$K_{abcd}(x, x'; t) = G_{abcd}(x') \Delta(x, x'; t) (1 + K(x, t)), \quad (9)$$

where

$$\Delta(x, x'; t) = \frac{\exp\left(-\frac{1}{4t} N_{ab}(x) (x - x')^a (x - x')^b\right)}{4\pi t^{3/2}}$$

and  $K(x, t)$  is a power series in  $t$  vanishing at  $t = 0$ . Using the fact that  $t$  has dimension  $m^{-2}$  we make the following expansion for  $K$  and  $N_{ab}$

$$K(x, t) = a_0 t R + (a_1 R^2 + b_1 R_{ab} R^{ab}) + \dots, \quad (10)$$

$$N_{ab}(x) = h_{ab} + 2t(A_0 R_{ab} + B_0 h_{ab} R) + t^2(B_1 R_a^c R_{cb} + A_1 R_{ab} R + C_1 h_{ab} R^2) + \dots, \quad (11)$$

where  $\dots$  denote the higher order terms which will not concern us, and  $a, b, A, B, C$  are the free parameters to be fixed.

Next we must resolve the ordering ambiguity in the operator  $\mathcal{H}$ . To this end we add the new term  $L_{ab}(x) \frac{\delta}{\delta h_{ab}(x)}$ , where

$$L_{ab} = \alpha h_{ab} + \beta h_{ab} R + \gamma R_{ab} + \dots \quad (12)$$

contains free coefficients do be fixed along with the coefficients in  $K$  and  $N_{ab}$ . Thus the final form of the Wheeler–De Witt operator is<sup>1</sup>

$$\begin{aligned} \mathcal{H}(x) = & \kappa^2 \int dx' K_{abcd}(x, x'; t) \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} + \\ & + \kappa^2 L_{ab}(x) \frac{\delta}{\delta h_{ab}(x)} + \frac{1}{\kappa^2} \sqrt{h} (R + 2\Lambda). \end{aligned} \quad (13)$$

To set the stage, we still need to define the action of operators on states. To this end we must discuss the issue of regularisation and renormalization. The operator (13) acting on a state (defined as an integral of a scalar density) produces, in general, terms with arbitrary (positive and negative) powers of  $t$ . This provides the regularised version of the operator since all the terms are finite, and singularities of the form  $\delta(0)$ ,  $\delta'(0)$ , etc. are traded for terms which are singular for  $t \rightarrow 0$ . Observe that the singular part of the action of

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<sup>1</sup>In the paper [9] we took  $\tilde{K}_{abcd}(x, x') = G_{abcd}(x) \tilde{K}(x, x')$ , where  $\tilde{K}$  was a heat kernel, and  $L_{ab}$  was taken to be the functional derivative of  $\tilde{K}_{abcd}$  with respect to  $h_{cd}$ .

the operator on a state depends on this state. To renormalize, we follow the procedure proposed by Mansfield [10], based on analytic continuation, which result in the following: the terms with positive powers of  $t$  are dropped, and the singular terms of the form  $\frac{1}{(4\pi)^{3/2}}t^{-k/2}$  are replaced by the renormalization coefficients  $\rho^k$ . Thus we are given a finite action of the Wheeler–De Witt operator on any state.

There is a number of important comments that must be made at this point. It is easy to see that the singular part of the regularised action, and thus the renormalized action of the Wheeler–De Witt operator on a state does depend on the state (cf. (A.1) and (A.2) in the Appendix.) This is clearly a natural feature of any regularisation technique. Thus, as already stressed above, the condition for anomaly cancellation should be analysed state-by-state.

Next, the coefficients in the regulator  $K$  are metric dependent. This should not be understood as an indication that the regulator depends on a background metric. It was observed by many that a wonderful feature of the metric formulation is that the metric (and its derivatives) appearing in the commutator is to be understood as result of action of the metric operator on a state. But in the metric representation the metric acts by multiplication ( $\hat{h}_{ab}(x)|* > = h_{ab}(x)|* >$ ), and thus we can always use  $h_{ab}(x)$  instead of  $\hat{h}_{ab}(x)$ . Also, the regulator seemingly depends on a background structure through the presence of the explicit  $x^a$  terms in the exponents. Such terms are necessarily present in any regulator based on point splitting technique. However it will be shown below that there is no anomaly in the quantum mechanical commutator of hamiltonian and diffeomorphism constraints and this means that the background structure dependence disappears in the final results.

Now we can turn to the interpretation of equation (8). According to our general philosophy explained above, we understand it in the following way. A constraint operator acts on a state and after renormalization gives another state depending on renormalization constants and the parameters of the regulator. On this resulting state the second operator acts. Thus the formal relation (7) is defined to mean (the state  $\Psi$  is, by definition, diffeomorphism-invariant)

$$\mathcal{D}(\mathcal{H}\Psi)_{ren} \sim (\mathcal{H}\Psi)_{ren}, \quad (14)$$

and, similarly, for the hamiltonian–hamiltonian commutator

$$(\mathcal{H}[N] (\mathcal{H}[M]\Psi)_{ren})_{ren} - (\mathcal{H}[M] (\mathcal{H}[N]\Psi)_{ren})_{ren} = 0 \quad (15)$$

for all  $M$  and  $N$ . In the formula above we used the smeared form of the Wheeler–De Witt operator

$$\mathcal{H}[M] = \int dx M(x) \mathcal{H}(x).$$

Let us turn back to equation (14). Since the action of diffeomorphism is standard, it suffices to check that  $(\mathcal{H}\Psi)_{ren}$  is a scalar density. But this is clearly the case: the first functional derivative acting on a state produces a tensor density  $\mathsf{T}^{ab}(x')$ . After acting by the second derivative and contracting indices, we obtain the terms of the form

$$\mathsf{T}_0(x')\delta(x', x) + \mathsf{T}_1(x') \circ \nabla^{x'} \circ \nabla^{x'} \delta(x', x) + \mathsf{T}_2(x') \circ \nabla^{x'} \circ \nabla^{x'} \circ \nabla^{x'} \circ \nabla^{x'} \delta(x', x) + \dots$$

where  $\circ$  denotes various indices contractions, and  $\mathsf{T}_n$  are tensor densities. These terms are multiplied by  $\Delta(x, x'; t)$  and integrated over  $x'$ . Now we integrate by parts which results in replacing covariant derivatives acting on  $K$  with appropriate powers of  $t^{-1}$  multiplied by some coefficients. After renormalization we obtain a scalar density as required. The action of the  $L$  term clearly gives the same result. Thus

*For the states being integrals of scalar densities there is no anomaly in the diffeomorphism — hamiltonian commutator*

This result is quite important because the anomaly in the string theory appears in the diffeomorphism — hamiltonian commutator. It proves also that, in spite of implicit co-ordinate system present in the construction of the regulator, the three dimensional diffeomorphisms are not broken by quantum corrections.

Now we turn to the most complicated problem, the hamiltonian — hamiltonian commutator (15). Our goal will be to use this equation to partially fix the free coefficients in  $K$ ,  $N_{ab}$ ,  $L_{ab}$ . These coefficients will be further fixed by demanding existence of solutions of Wheeler–De Witt equation. In what follows we will be interested in solutions of the form  $\Psi(\mathcal{V}, \mathcal{R})$ . Therefore we

check explicitly the closure of the algebra only for the states of this form. We will comment on the general case at the end of this section.

Let us start with the simplest state  $\Psi = 1$ . The action of the first smeared operator gives simply

$$(\mathcal{H}[M]\Psi)_{ren} = \frac{1}{\kappa^2} \int dx \sqrt{h}(x) M(x) (R(x) + 2\Lambda). \quad (16)$$

Now we have to act on the right hand side of the above equation with the operator  $(\mathcal{H}[N]$ , then renormalize the result, and finally subtract the result of the same calculation with  $N$  interchanged with  $M$ . After rather tedious computation one finds in the commutator the term proportional to  $[N, M]^a = N\nabla^a M - M\nabla^a M$  which must vanish, to wit

$$\rho^{(1)} a_0 \frac{3}{2} \nabla_a R - (\nabla_a L - \nabla_b L_a^b) = 0, \quad (17)$$

where  $L = h_{ab} L^{ab}$ . Using Bianchi identity and the expansion (12) we find the first relation between coefficients, to wit (it will soon turn out that ... terms in  $L_{ab}$  vanish)

$$\frac{1}{2} (3\rho^{(1)} a_0 - \gamma) - 2\beta = 0. \quad (18)$$

Now let us turn to the states depending of  $\mathcal{V} = \int_M d^3x \sqrt{h}$ . Let  $\mathcal{H}[M]$  act on this state. From Eq. (A.1) we see that we have an equation which is of the form (16) (with different coefficients which will include  $\mathcal{V}$ .) Therefore the condition for vanishing of the commutator is the same as above, (18).

Let us pause for a moment with investigation of the algebra to make an important observation. We want some  $\Psi(\mathcal{V})$  to be a solution of the Wheeler–De Witt equation. From the computations above we see that the double derivative term in  $\mathcal{H}$  will produce terms up to order  $R$ . It follows that, while solving the equation, we would not be able to cancel terms of higher order in  $R$  (like  $R^2$ .) Therefore, all the terms in  $L_{ab}$  expansion (12) denoted by ... must vanish. Thus we take

$$L_{ab} = \alpha h_{ab} + (\gamma R_{ab} + \beta h_{ab} R),$$

where the coefficients  $\beta$  and  $\gamma$  are subject to the condition (18).

Now we turn to the wavefunction  $\Psi = \Psi(\mathcal{R})$ . Let us analyse the action of the commutator of hamiltonian in a number of steps. The first observation

follows from the  $\Psi''$  term in (A.2). It can be checked that after acting on this term by  $\mathcal{H}[N]$  one obtains a term proportional to  $\Psi'''$  which contains unremovable anomaly of the form

$$[N, M]^a \left( \frac{3}{8} R_{ab} \nabla^b R - \frac{3}{16} R \nabla_a R \right).$$

Till this point we assumed that the wavefunction  $\Psi(\mathcal{R})$  was arbitrary, thus anomaly multiplying  $\Psi'''$  was to vanish independently of possible anomalies multiplying different derivatives of  $\Psi$ . But this is, clearly, cannot be accomplished. We thus have no choice but to restrict  $\Psi$ . It would seem that fixing the background geometry

$$-\frac{3}{8} R^2 + R_{ab} R^{ab} = 0 \tag{19}$$

would do, but this cannot be done because some external condition may be applied only after the commutator is computed, and not at the first step. The only way out is to make  $\Psi'' = 0$ , or proportional to  $\Psi'$  (where the terms quadratic in curvature are already present.) This means that either  $\Psi(\mathcal{R}) = A\mathcal{R}$  or  $\Psi(\mathcal{R})$  is a linear combinations of exponents  $\exp(\omega\mathcal{R})$ . It should be stressed that since solutions we are after will necessarily have the form of exponents, the restriction we are making is not as severe as it would seem at the first sight<sup>2</sup>. The inspection of equation (A.2) clearly shows that the most economic way is to take  $\mathcal{B} = -\frac{3}{8}\mathcal{J}$ . In this way we can cancel the anomaly for arbitrary  $\omega$ .

It turns out that the anomaly is proportional to  $[N, M]^a$  times a combination of five different tensorial objects. To cancel the anomaly proportional to  $R_c^b \nabla_b R_a^c$  we must put  $B_1 = 0$ . Similarly, the condition for the anomaly proportional to  $R_c^b \nabla_b R_a^c$  to vanish is  $b_1 = 0$ . From the conditions for  $R_a^b \nabla_b R$  and  $(\nabla_a R)R$ , and the equation  $\mathcal{B} = -\frac{3}{8}\mathcal{J}$  we find expressions for  $A_1$ ,  $C_1$ , and  $a_1$ , to wit

$$A_1 = -a_0 A_0 + \frac{11}{4} a_0 + \frac{1}{\rho^{(1)}} \left( \frac{5}{4} \gamma + \beta \right), \tag{20}$$

$$C_1 = -\frac{9}{8} a_0 A_0 - \frac{29}{16} a_0 - \frac{7}{8} \tag{21}$$

$$a_1 = -\frac{29}{66} a_0 - \frac{9}{11} a_0 A_0 - \frac{1}{\rho^{(1)}} \left( \frac{14}{11} \beta + \frac{37}{33} \gamma \right) \tag{22}$$

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<sup>2</sup>It is sufficient to check that there is no anomaly in a neighbourhood of solutions.

There is one equation remaining, being a coefficient of  $\nabla_a R$  anomaly which relates  ${}_0$  parameters to each other:

$$\rho^{(3)} \left( -\frac{3}{16} - \frac{1}{2}A_0 - \frac{11}{8}B_0 + \frac{1}{2}a_0 \right) - \frac{3}{2}\alpha = 0. \quad (23)$$

We are left therefore with six free coefficients of the regularised Wheeler–De Witt operator  $A_0$ ,  $B_0$ ,  $a_0$ ,  $\alpha$ ,  $\beta$ , and  $\gamma$  subject to two linear equations (18) and (23).

Thus the final form of the regularised Wheeler–De Witt operator which preserves the constraint algebra is (to linear order in  $R$  and with included)

$$\begin{aligned} \mathcal{H}(x) = & \kappa^2 \int dx' G_{abcd}(x') \Delta(x, x'; t) (1 + a_0 t R + \dots) \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} + \\ & + \kappa^2 (\alpha h_{ab} + (\beta h_{ab} R + \gamma R_{ab})) (x) \frac{\delta}{\delta h_{ab}(x)} + \frac{1}{\kappa^2} \sqrt{h} (R + 2\Lambda). \end{aligned} \quad (24)$$

The formula (24) completes our construction of the Wheeler–De Witt operator. As compared to the choice made in the paper [9], where we used the heat kernel and  $L_{ab}$  was its functional derivative, here we gained much more freedom in the form of additional free constants. These constant will be further fixed by demanding that the Wheeler–De Witt equations possesses a maximal number of solutions, that is that there are solutions  $\Psi(\mathcal{V})$ ,  $\Psi(\mathcal{R})$ , and  $\Psi(\mathcal{V}, \mathcal{R})$ .

It is possible to extend the above analysis to the states being functionals of higher powers of curvatures. To this end one has to add terms of order  $t^3$  to the regulator  $K$ , compute the commutator, and fix the coefficients as it was done above. It seems quite likely that the resulting equations could be solved. However, the computations are becoming extremely tedious, and for that reason in this paper we were not able to address the question of anomalies for higher states.

### 3 Solutions

From the previous section we know that the most general form of the Wheeler–De Witt operator satisfying our criteria is given by equation (24). Now, employing this operator, we will try to find a class of solutions of the Wheeler–De

Witt equation. It should be stressed at this point that we regard the existence of a maximal possible space of solutions as an ultimate condition fixing the operator completely. The reason for is that any regulator defines a quantum theory with a certain set of solutions. Clearly, a theory with richest set of solutions is most interesting. Thus our goal is twofold: to find solutions and to fix the operator as to allow for the maximal possible number of them.

We will consider the states of the form  $\Psi = \Psi(\mathcal{V}, \mathcal{R})$ . It is clear from the form of the Wheeler–De Witt equation that the resulting equations, multiplying various scalar functions will be linear, therefore, without loss of generality, we can assume that  $\Psi(\mathcal{V}, \mathcal{R}) = \exp(\sigma\mathcal{V} + \omega\mathcal{R})$ .

First we solve the equation for the coefficient  $\omega$ . From the part of the equation involving the square curvature terms, (A.2), we see easily that since  $\mathcal{B} = -\frac{3}{8}\mathcal{J}$ ,  $\omega = -\mathcal{J} = \gamma$ . Let us then turn to the coefficients multiplying  $\sqrt{h}R$ <sup>3</sup>

$$\kappa^2 \left( \sigma\mathcal{X} + \omega\mathcal{Y} - \frac{1}{4}\omega\sigma \right) + \frac{1}{\kappa^2} = 0, \quad (25)$$

where

$$\begin{aligned} \mathcal{X} &= -\frac{21}{8}a_0\rho^{(1)} + \frac{3}{2}\beta + \frac{1}{2}\gamma, \\ \mathcal{Y} &= \rho^{(3)} \left( A_0 + 3B_0 - \frac{3}{2}a_0 + \frac{7}{8} \right) + \frac{1}{2}\alpha. \end{aligned}$$

The coefficient multiplying the  $\sqrt{h}$  term reads

$$\kappa^2 \left( -\frac{3}{8}\sigma^2 - \frac{21}{8}\sigma\rho^{(3)} + \frac{3}{2}\sigma\alpha - \frac{3}{2}\rho^{(5)}\omega \right) + \frac{2\Lambda}{\kappa^2} = 0. \quad (26)$$

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<sup>3</sup>Observe that in addition to terms presented in appendix, (A.2), (A.1), there is another term resulting from the action of second functional derivative, one on  $\mathcal{V}$ , and one on  $\mathcal{R}$ .

*Solutions with  $\Lambda \neq 0$*

We consider three cases:

*Case I.*  $\omega = 0$ . In this case we have

$$\Psi = \exp(\tilde{\sigma}\mathcal{V}), \quad \tilde{\sigma} = -\frac{1}{\kappa^4\mathcal{X}},$$

and equation (26) gives a condition for the parameters of the regulator which can be solved for  $\alpha$ .

*Case II.*  $\omega = \gamma$  and  $\sigma = 0$ . We have

$$\mathcal{Y} = -\frac{1}{\kappa^4\gamma}$$

and the condition relating  $\gamma$  to the bare coupling and renormalization constants

$$\gamma = \frac{4\Lambda}{3\kappa^4} \frac{1}{\rho^{(5)}}. \quad (27)$$

*Case III*  $\omega = \gamma$  and  $\sigma \neq 0$ . We find  $\sigma = 4\alpha - 7\rho^{(3)}$  and the condition  $\mathcal{X} = \frac{1}{4}\gamma$ . This condition can be solved along with conditions from the previous section to give expressions for the regulator parameters.

Collecting all results we finally have

$$\Psi_I = \exp\left(-3\frac{\rho^{(5)}}{\Lambda}\mathcal{V}\right) \quad \text{Case I;} \quad (28)$$

$$\Psi_{II} = \exp\left(\frac{4\Lambda}{3\kappa^4\rho^{(5)}}\mathcal{R}\right) \quad \text{Case II;} \quad (29)$$

$$\Psi_{III} = \exp\left(-3\frac{\rho^{(5)}}{\Lambda}\mathcal{V} + \frac{4\Lambda}{3\kappa^4\rho^{(5)}}\mathcal{R}\right) \quad \text{Case III.} \quad (30)$$

Observe that solution III is a product of solutions I and II. We will return to this observation below.

*Solutions with  $\Lambda = 0$*

it is easy to see that (assuming  $\rho^{(5)} \neq 0$ ) in this case the wavefunction  $\Psi = \exp(\gamma\mathcal{R})$  does not solve the Wheeler–De Witt equation. Taking  $\mathcal{X} = \frac{1}{4}\gamma$  as above we find the solution

$$\Psi_{I_0} = \exp\left(-\frac{4}{\kappa^4\gamma}\mathcal{V}\right) \quad (31)$$

The second solution is of the form

$$\Psi_{III_0} = \exp(\sigma\mathcal{V} + \gamma\mathcal{R}), \quad (32)$$

where  $\sigma$  is a solution of the following quadratic equation

$$\sigma^2 + \frac{4}{\kappa^4\gamma}\sigma - 4\rho^{(5)}\gamma = 0. \quad (33)$$

Depending on the value of

$$\left(\frac{1}{\kappa^4\gamma}\right)^2 + \rho^{(5)}\gamma$$

we have either two real, or two complex, or one real solution.

Thus we have three different wavefunctions (for both cases  $\Lambda = 0$  and  $\Lambda \neq 0$ ) being solutions of the Wheeler–De Witt equation and containing functionals of order at most linear in  $R$ . It is very interesting that the solutions depend on the bare coupling constants  $\kappa$  and  $\Lambda$  and only on a single renormalization constant  $\rho^{(5)}$ . Of course, any linear combination (with complex coefficients) of the solution is a solution. Such combinations will be called below “Schrödinger cat universes”. It can be argued that, contrary to the real solutions, complex solutions will in general possess a nontrivial time evolution.

## A Renormalized action of $\mathcal{H}[M]$

Here we present the calculation of the renormalized action of hamiltonian constraint on states. We have

$$G_{abcd}(x') \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} \mathcal{V} = -\frac{21}{8} \delta(x - x').$$

Thus

$$\int d^3x' K(x, x'; t) G_{abcd}(x') \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} \mathcal{V} =$$

$$-\sqrt{h} \frac{21}{8} \frac{1}{(4\pi)^{3/2}} \left( \frac{1}{t^{3/2}} + \frac{1}{t^{1/2}} a_0 R + O(t) \right).$$

Using this result and renormalizing, we obtain

$$\mathcal{H}[M] \Psi(\mathcal{V}) = \int d^3x \sqrt{h} M \left\{ -\kappa^2 \frac{3}{8} \Psi''(\mathcal{V}) + \right.$$

$$\left. + \kappa^2 \Psi'(\mathcal{V}) \left( -\frac{21}{8} (\rho^{(3)} + a_0 \rho^{(1)} R) + \frac{1}{2} L \right) + \frac{1}{\kappa^2} (R + 2\Lambda) \Psi(\mathcal{V}) \right\}, \quad (\text{A.1})$$

where  $L = L_{ab} h^{ab}$ .

Similarly,

$$G_{abcd}(x') \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} \mathcal{R} = \frac{7}{8} R(x') \delta(x - x') + \square_{x'} \delta(x - x').$$

Thus

$$\int d^3x' K(x, x'; t) G_{abcd}(x') \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} \mathcal{R} =$$

$$\sqrt{h} \left[ -\frac{3}{2} \rho^{(5)} + \rho^{(3)} R \left( A_0 + 3B_0 - \frac{3}{2} a_0 + \frac{7}{8} \right) + \right.$$

$$\left. + \rho^{(1)} \left\{ R^2 \left( \frac{7}{8} a_0 + A_1 + 3C_1 - \frac{3}{2} a_1 + (A_0 + 3B_0) a_0 \right) + \right. \right.$$

$$\left. \left. + R_{ab} R^{ab} \left( B_1 - \frac{3}{2} b_1 + 3D_1 \right) \right\} \right].$$

Thus we obtain

$$\mathcal{H}[M] \Psi(\mathcal{R}) = \int d^3x \sqrt{h} M \left\{ \kappa^2 \Psi''(\mathcal{R}) \left( -\frac{3}{8} R^2 + R_{ab} R^{ab} \right) + \right.$$

$$\left. + \kappa^2 \Psi'(\mathcal{R}) \left[ \mathcal{B} R^2 + \mathcal{J} R_{ab} R^{ab} + R \left( \rho^{(3)} \left( A_0 + 3B_0 - \frac{3}{2} a_0 + \frac{7}{8} \right) + \frac{1}{2} \alpha \right) - \frac{3}{2} \rho^{(5)} \right] + \right.$$

$$\left. \frac{1}{\kappa^2} (R + 2\Lambda) \Psi(\mathcal{R}) \right\}, \quad (\text{A.2})$$

where

$$\mathcal{B} = \left[ \frac{7}{8}a_0 - \frac{3}{2}a_1 + (A_0 + 3B_0)a_0 + A_1 + 3C_1 \right] \rho^{(1)} + \frac{1}{2}(\beta + \gamma), \quad (\text{A.3})$$

and

$$\mathcal{J} = \left( -\frac{3}{2}b_1 + B_1 + 3D_1 \right) \rho^{(1)} - \gamma. \quad (\text{A.4})$$

Equations (A.1) and (A.2) are basic for our investigations in the main body of the paper. The expressions in the parentheses  $\{*\}$  in these equations are the Wheeler–De Witt equations for the corresponding wavefunctions.

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